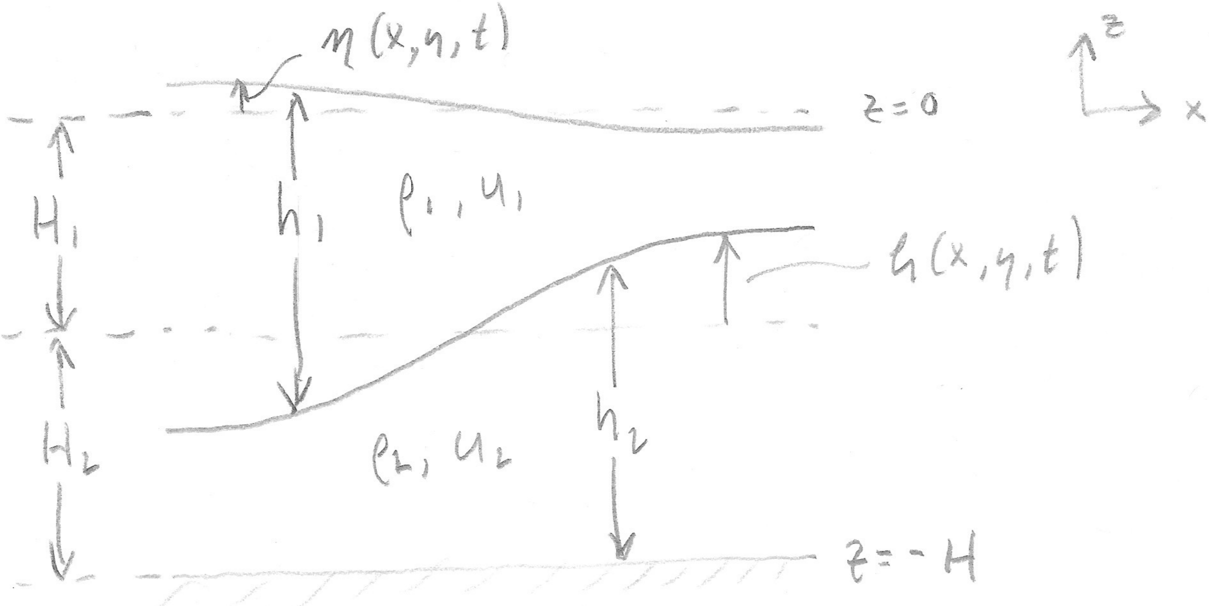
Stratification : 2 Layers

7/18/2019

- internal waves (this lecture)
- river plumes
- Estuaries : Knudsen's Relation & Salt Wedge
- Hydraulic exchange at sill
- Coasts : Internal Kelvin Wave, Upwelling



- Density change small $\rho_2 - \rho_1 \equiv \Delta\rho \ll \rho_0$

  1-10 : 1000

- But internal displacements large $[\zeta] \gg [\eta]$

$\Rightarrow$ p_x due to ζ_x comparable to those from η_x

2 Layer internal waves

- math just like SW waves
- But phase speed much slower:

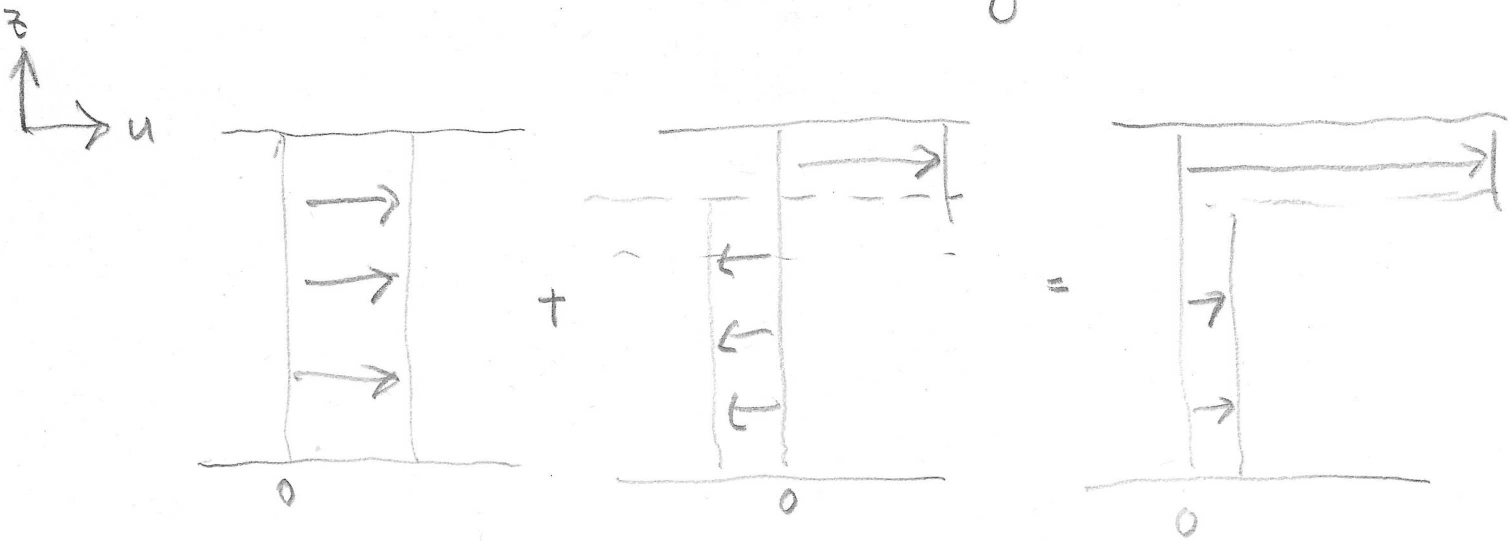
$$c = \sqrt{g' H_{eff}}$$

where $g' = \frac{g \Delta \rho}{\rho_0}$ Reduced Gravity

$$H_{eff} = \frac{H_1 H_2}{H_1 + H_2}$$ Effective Depth

- calculate g', H_{eff}, c for your region
- compare c & $c_{SW} = \sqrt{gH}$
- What is H_{eff} for $H_1 \ll H_2$?

Linear waves are a sum of "modes"



External	Internal	Total
Barotropic	1st Baroclinic Mode	But IW are
SW Waves	$\int u \, dz = 0$	slower w/ shorter wavelength

2 Layer Pressure Gradient:

Hydrostatic $\quad p_z = -\rho g \rightarrow p = \cancel{p_{atm}} + g\int_z^{\eta} \rho \, dz$

ignore

$$p_1 = \rho_1 g(\eta - z)$$

$$p_2 = \rho_1 g(h_1) + \rho_2 g(h - z - H_1)$$

$\eta + H_1 - h$

$$\therefore \quad p_{1x} = \rho_1 g \eta_x$$

$$p_{2x} = \rho_1 g \eta_x - \rho_1 g h_x + \rho_2 g h_x$$

$$= \rho_1 g \eta_x + g \Delta\rho\, h_x$$

then taking $\frac{\rho_1}{\rho_0} \sim 1$ and defining $g' = \frac{g \Delta\rho}{\rho_0}$

$$\boxed{\begin{aligned} -\frac{1}{\rho_0} p_{1x} &= -g\eta_x \\ -\frac{1}{\rho_0} p_{2x} &= -g\eta_x - g' h_x \end{aligned}}$$

So our equations are:

$$\text{mass}_1 \quad -h_t + H_1 u_{1x} = 0$$
← assumed $[\eta] \ll [h]$

also assumed $[h] \ll H_1 + H_2$

$$\text{mass}_2 \quad h_t + H_2 u_{2x} = 0$$

$$\text{mom}_1 \quad u_{1t} + g\eta_x = 0$$
← η still important here

$$\text{mom}_2 \quad u_{2t} + g\eta_x + g' h_x = 0$$
Why?

Key assumption: to get $\int u \, dx = 0$

shape of η & h should be similar

so assume $\quad \eta = \alpha h \quad (+)$

and solve for α that gives $\int u \, dz = 0$

→ rewrite mom eqns: using $(+)$

$$\text{mom}_1 \quad u_{1t} + g\alpha h_x = 0$$

$$\text{mom}_2 \quad u_{2t} + (g\alpha + g')h_x = 0$$

$\boxed{\text{mass}_1 + \text{mass}_2}_t \Rightarrow (H_1 u_1 + H_2 u_2)_{xt} = 0$

(and $H_1 u_1 + H_2 u_2 = 0$)

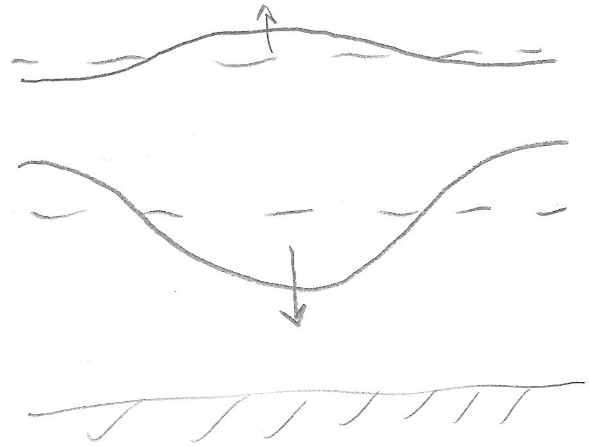
so $H_1 \boxed{\text{mom}_1}_x + H_2 \boxed{\text{mom}_2}_x$ gives

$$\left[g\alpha H_1 + H_2 (g\alpha + g') \right] \eta_{xx} = 0$$

$\Rightarrow$ Required value of α is

$$\boxed{\alpha = \frac{-H_2}{H_1 + H_2} \frac{g'}{g}}$$

$[\alpha] \ll 1$ and negative



Finally, get wave eqn. from

Layer 1 equations:

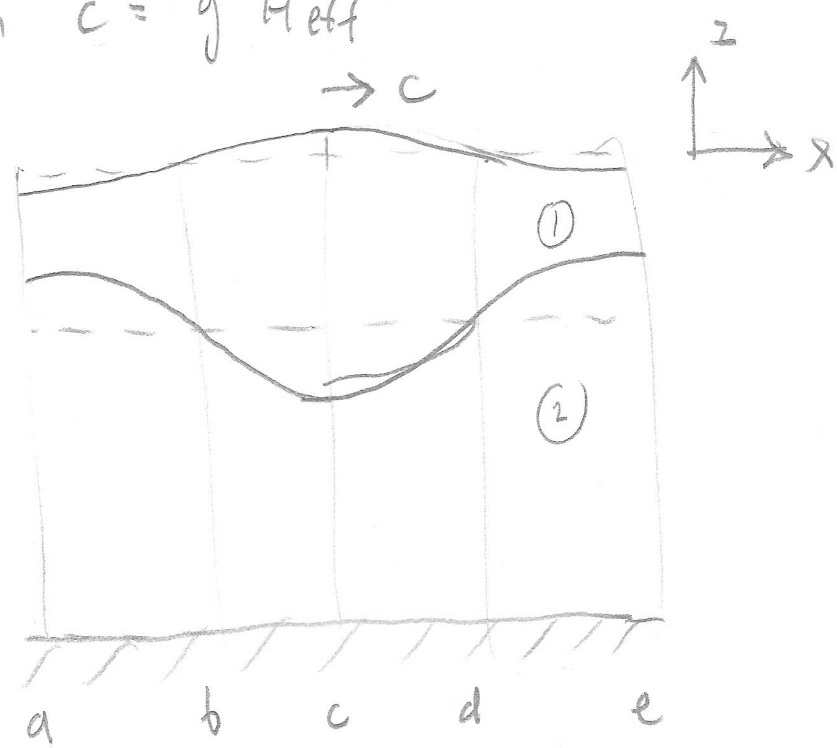
$$H_1 \frac{\partial}{\partial x}\left[u_{1t} + g\alpha \eta_x = 0 \right] \rightarrow H_1 u_{1xt} - g' \frac{H_1 H_2}{H_1 + H_2} \eta_{xx} = 0$$

add $\Updownarrow$

$$-\frac{\partial}{\partial t}\left[-\eta_t + H_1 u_{1x} = 0 \right] \rightarrow -H_1 u_{1xt} + \eta_{tt} = 0$$

$$\Rightarrow \boxed{\eta_{tt} - g' H_{eff}\, \eta_{xx} = 0}$$

wave equation with $c^2 = g' H_{eff}$



For this wave:

Draw velocity in both layers at a–e: